INTRINSICALLY n-LINKED COMPLETE BIPARTITE GRAPHS

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ABSTRACT. We prove that every embedding of $K_{2n+1,2n+1}$ into $\mathbb{R}^3$ contains a non-split link of n-components. Further, given an embedding of $K_{2n+1,2n+1}$ in $\mathbb{R}^3$, every edge of $K_{2n+1,2n+1}$ is contained in a non-split n-component link in $K_{2n+1,2n+1}$.

1. Introduction

A graph, G is *intrinsically linked* if every embedding of G into $\mathbb{R}^3$ contains a nontrivial link. Conway and Gordon [2] and Sachs [7] first showed the existence of such graphs by proving the complete graph on six vertices, K_6, is intrinsically linked. Sachs [7] proved that the graphs in the Petersen family are intrinsically linked and that no minor of them is intrinsically linked. Then Robertson, Seymour, and Thomas [5] proved that any intrinsically linked graph contains a graph in the Petersen family as a minor. Together these results fully characterize intrinsically linked graphs.

The idea of intrinsically linked graphs can be generalized to a graph that intrinsically contains a link of more than two components. A link L is *split* if there is an embedding of a 2-sphere F in $\mathbb{R}^3 \smallsetminus L$ such that each component of $\mathbb{R}^3 \smallsetminus F$ contains at least one component of L. A graph G is *intrinsically n-linked* if every embedding of G into $\mathbb{R}^3$ contains a non-split n-component link. Flapan, Naimi, and Pommersheim investigate intrinsically triple linked graphs in [4]. They proved that K_{10} is the smallest complete graph to be intrinsically triple linked. The more general question of what the smallest m is such that K_m is intrinsically n-linked is still open. Bowlin and Foisy [1] also look at intrinsically triple linked graphs. They exhibit two different subgraphs of K_{10} that are also intrinsically triple linked, proving that K_{10} is not minor minimal with respect to being intrinsically triple linked. However, it is not known if either of these subgraphs is minor minimal. Flapan, Foisy, Naimi, and Pommersheim address the question of minor minimal intrinsically n-linked graphs in [3], where they construct families of minor minimal intrinsically n-linked graphs.

In this paper we consider the question, for a given n, what is the smallest r and s such that the complete bipartite graph $K_{r,s}$ is intrinsically n-linked. We prove that $K_{2n+1,2n+1}$ is intrinsically n-linked. A similar argument shows that the complete tripartite graph $K_{2n,2n,1}$ is intrinsically n-linked.

Though it is not known what the smallest m is such that K_m is intrinsically n-linked, the number of vertices m can be bounded. First based on the number of disjoint simple closed curves needed in K_m we obtain a lower bound of $3n$. This lower bound is realized in the $n = 2$ case, but no longer for the $n = 3$ case, where $m = 10$ [4]. It follows from [3] that K_{7n-6} is intrinsically n-linked. Since $K_{2n,2n,1}$

is a subgraph of $K_{2n+2n+1}$ Theorem 2 gives an upper bound for m of $4n+1$. This is a significant improvement over the earlier bound of $7n-6$.

2. Key Lemmas

Let a simple closed curve containing exactly four vertices be called a *square*. Let $\omega(J,K) = lk(J,K) \pmod 2$. Hence $\omega(J,K)$ is the number of times J crosses over $K \pmod 2$. When two simple closed curves, J and K are said to link, it should be understood that this means $\omega(J,K) \neq 0$. Let $K_{r,s}$ be embedded in $\mathbb{R}^3$, and let γ be a simple closed curve in $\mathbb{R}^3 \smallsetminus K_{r,s}$. We say γ *links* $K_{r,s}$ if there exists a square J in $K_{r,s}$ such that $\omega(\gamma, J) \neq 0$. The graph $K_{3,3}$ has nine edges, and each edge is contained in four squares. Label by {1,3,5} and {2,4,6} the two sets of three vertices of $K_{3,3}$. Each square can be uniquely denoted by a four-number sequence, starting with an odd, and alternating odd and even with the odds and evens each put in increasing order.

Lemma 1. *Let $K_{3,3}$ be embedded in $\mathbb{R}^3$, and let γ be a projection of a simple closed curve in $\mathbb{R}^3 \smallsetminus K_{3,3}$. Let γ_o denote γ with one of the over-crossings in an edge f changed to an under-crossing. Suppose γ_o links $K_{3,3}$ in zero squares. Then γ links $K_{3,3}$ in four squares, all containing the edge f.*

Proof. A square J in $K_{3,3}$ links γ if $\omega(\gamma, J) \neq 0$. Here

$$\omega(\gamma, J) = \omega(\gamma_o, J) + \begin{cases} 1, & \text{if } f \in J \\ 0, & \text{if } f \notin J \end{cases} \pmod 2.$$

So

$$\omega(\gamma, J) = 0 + \begin{cases} 1, & \text{if } f \in J \\ 0, & \text{if } f \notin J \end{cases} \pmod 2 = \begin{cases} 1, & \text{if } f \in J \\ 0, & \text{if } f \notin J \end{cases} \pmod 2.$$

The edge f, like any edge, is contained in four squares in $K_{3,3}$. Thus γ links $K_{3,3}$ in four squares all of which contain the edge f. □

Lemma 2. *Let $K_{3,3}$ be embedded in $\mathbb{R}^3$, and let γ be a projection of a simple closed curve in $\mathbb{R}^3 \smallsetminus K_{3,3}$. Let γ_o denote γ with one of the over-crossings in an edge f changed to an under-crossing. Suppose γ_o links $K_{3,3}$ in four squares, all of which contain the edge e. Then:*

- *if $f = e$, γ links $K_{3,3}$ in zero squares*
- *if f is adjacent to e, γ links $K_{3,3}$ in four squares*
- *if f is nonadjacent to e, γ links $K_{3,3}$ in six squares.*

Proof. Again, if J is a square in $K_{3,3}$, then

$$\omega(\gamma, J) = \omega(\gamma_o, J) + \begin{cases} 1, & \text{if } f \in J \\ 0, & \text{if } f \notin J \end{cases} \pmod 2.$$

Here,

$$\omega(\gamma_o, J) = \begin{cases} 1, & \text{if } e \in J \\ 0, & \text{if } e \notin J \end{cases} \pmod 2.$$

So if $e = f$,

$$\omega(\gamma, J) = \begin{cases} 1+1, & \text{if } e = f \in J \\ 0+0, & \text{if } e = f \notin J \end{cases} \pmod 2 = 0 \pmod 2.$$

So γ links $K_{3,3}$ in zero squares.

Next, if f is adjacent to e, then it appears in two of the four squares containing e. Say, J_1, J_2, J_3 and J_4 are the squares that contain e. Let J_3, J_4, J_5 and J_6 be the squares that contain f. If J is a square that contains neither e nor f, then $\omega(\gamma, J) = \omega(\gamma_o, J) = 0$. So we need only consider the above squares J_i. Now,

$$\omega(\gamma, J_i) = 1 + 0 \pmod 2 = 1 \pmod 2, \text{ if } i = 1, 2,$$

$$\omega(\gamma, J_i) = 1 + 1 \pmod 2 = 0 \pmod 2, \text{ if } i = 3, 4,$$

and

$$\omega(\gamma, J_i) = 0 + 1 \pmod 2 = 1 \pmod 2, \text{ if } i = 5, 6.$$

So γ links $K_{3,3}$ in the four squares J_1, J_2, J_5, and J_6.

Finally, if f is nonadjacent to e then it appears in exactly one of the four squares containing e. Let J_1, J_2, J_3, and J_4 be the squares that contain e, and let J_4, J_5, J_6, and J_7 be the squares that contain f. By an argument similar to the above, we see that γ links $K_{3,3}$ in the six squares J_1, J_2, J_3, J_5, J_6, and J_7. $\square$

Lemma 3. *Let $K_{3,3}$ be embedded in $\mathbb{R}^3$, and let γ be a projection of a simple closed curve in $\mathbb{R}^3 \setminus K_{3,3}$. Let γ_o denote γ with one of the over-crossings in an edge f changed to an under-crossing. Suppose γ_o links $K_{3,3}$ in six squares. Then:*

- *if f appears in three of the squares that γ_o links, γ links $K_{3,3}$ in four squares*
- *if f appears in two of the squares that γ_o links, γ links $K_{3,3}$ in six squares.*

Proof. Let J_1, J_2, J_3, J_4, J_5, and J_6 be the squares that γ_o links in $K_{3,3}$. Suppose f appears in three of these squares. Let f be contained in J_4, J_5, J_6, and J_7. Similar to the proof of Lemma 2, γ links $K_{3,3}$ in the four squares J_1, J_2, J_3, and J_7.

Next, suppose f appears in two of these squares. Let f be contained in J_5, J_6, J_7, and J_8. Then γ links $K_{3,3}$ in the six squares J_1, J_2, J_3, J_4, J_7, and J_8. $\square$

We use the above lemmas to prove the following lemma.

Lemma 4. *Let $K_{3,3}$ be embedded in $\mathbb{R}^3$, and let γ be a simple closed curve in $\mathbb{R}^3 \setminus K_{3,3}$. Then one of the following holds:*

- **(a):** *γ links $K_{3,3}$ in zero squares.*
- **(b):** *γ links $K_{3,3}$ in four squares all of which contain a common edge.*
- **(c):** *γ links $K_{3,3}$ in six squares and there exists three mutually nonadjacent edges of $K_{3,3}$ which appear in only two of the six squares. (All other edges appear in precisely three of the six squares.)*

Proof. Fix a projection of γ and $K_{3,3}$. The proof is by induction on the number of over-crossings of γ with $K_{3,3}$.

Suppose $n = 1$. So γ has one over-crossing and it occurs on one edge of $K_{3,3}$, say e. If we change the over-crossing to an under-crossing, we obtain γ_o which links zero squares in $K_{3,3}$. So by Lemma 1, γ links all of the squares that contain e, of which there are four, and γ satisfies **(b)**.

Suppose that γ crosses over $K_{3,3}$ $n+1$ times. Define γ_o to be the simple closed curve γ with one of the over-crossings in edge f changed to an under-crossing. By our inductive hypothesis, γ_o satisfies **(a)**, **(b)**, or **(c)**. If γ_o satisfies **(a)** then by Lemma 1, γ satisfies **(b)**. If γ_o satisfies **(b)**, we may assume without loss of generality, γ_o links the squares 1234, 1236, 1254, and 1256, all containing the edge $\overline{12}$. Hence, from Lemma 2 we know that γ links zero, four, or six squares in $K_{3,3}$. If the edge f is adjacent to $\overline{12}$, without loss of generality $f = \overline{23}$. Then, by the

proof of Lemma 2, γ links the squares 1254, 1256, 3254, and 3256. So γ links four squares that all contain the edge $\overline{25}$ and thus it satisfies **(b)**. Next, if the edge f is nonadjacent to $\overline{12}$, without loss of generality $f = \overline{34}$. By the proof of Lemma 2, γ links the squares 1254, 1436, 1236, 3254, 1256,and 3456. In these six squares, the mutually nonadjacent edges $\overline{14}$, $\overline{32}$, and $\overline{56}$ each appears only twice (while all other edges of $K_{3,3}$ appear thrice). So the curve γ satisfies **(c)**.

If γ_o satisfies **(c)**, without loss of generality γ_o links the six squares where the edges $\overline{12}$, $\overline{34}$, and $\overline{56}$ appear only twice. Then γ_o links 1236, 1254, 1436, 3254, 1456, and 3256. Every edge of $K_{3,3}$ either appears in two or three of the squares that γ_o links. If f appears in three, without loss of generality $f = \overline{14}$. If f appears in two, without loss of generality $f = \overline{12}$. Recall that, γ links a square J if and only if either J contains f and J does not link γ_o, or J does not contain f and J links γ_o. If $f = \overline{14}$, then γ links the four squares 1234, 1236, 3254, and 3256, which all contain the edge $\overline{23}$. Hence γ satisfies **(b)**. If $f = \overline{12}$, then γ links the six squares 1234, 1256, 1436, 3256, 1456, and 3254, in which the three mutually nonadjacent edges $\overline{12}$, $\overline{36}$, and $\overline{54}$ appear only twice. Hence γ satisfies **(c)**. $\square$

From here forward, when it is said that γ links a graph isomorphic to $K_{3,3}$ in four squares, it should be understood that γ satisfies **(b)** in Lemma 4, as this is the only way γ can link $K_{3,3}$ in four squares. Similarly, if γ is said to link six squares in $K_{3,3}$, that should be taken to mean γ satisfies **(c)**.

3. Main results

We will use the following definition in proving the main result.

Definition 1. *Let M be a subgraph of $K_{4,4}$ which is isomorphic to $K_{3,3}$, and let α be a square in M. The* α-opposite subgraph of *M, say N, isomorphic to $K_{3,3}$ is the subgraph defined by the four vertices in α and the two vertices in $K_{4,4} \setminus M$. Then M and N are a* pair of α-opposite subgraphs *in $K_{4,4}$.*

Theorem 1. *Given $n > 1$, every embedding of the complete bipartite graph $K_{2n+1,2n+1}$ into $\mathbb{R}^3$ contains a non-split n-component link.*

Proof. We shall prove by induction on n that every embedding of $K_{2n+1,2n+1}$ contains a non-split n-component link L of squares, with a component J such that $L \setminus J$ is a non-split $(n-1)$-component link. When $n = 2$, $K_{2n+1,2n+1} = K_{5,5}$. The graph $K_{4,4} \subset K_{5,5}$ is known to be intrinsically 2-linked (by Sachs [7], $K_{4,4}$ with one edge removed is intrinsically linked). In this case, since $K_{4,4}$ contains a non-split 2-component link, both components must be squares and either component can be chosen to be J.

Consider an embedding of $K_{2n+1,2n+1}$ into $\mathbb{R}^3$. Fix a projection of the embedded $K_{2n+1,2n+1}$. It has a subgraph, say H, which is isomorphic to $K_{2(n-1)+1,2(n-1)+1}$. By the inductive hypothesis, H contains a non-split $(n-1)$-component link L of squares with a component J such that $L \setminus J$ is a non-split $(n-2)$-component link. Let H_1 be the subgraph of H that is defined by the vertices of L. Next choose a square γ in L that links J. Let the subgraph isomorphic to $K_{5,5}$, defined by the vertices of $(K_{2n+1,2n+1} \setminus H_1) \cup J$ be called G. Since γ is in H_1 and is disjoint from J, it is disjoint from G. The curve γ links J so, by Lemma 4, γ links each of the $K_{3,3}$ subgraphs containing J in four or six squares.

We will consider two different cases which each break into two subcases and show in each case that there are two disjoint squares in G that either both link γ or they

are linked together and one of them links γ. This will finish the proof by finding the desired non-split n-link in the given embedding of $K_{2n+1,2n+1}$.

Case 1: The curve γ does not link any of the $K_{3,3}$ subgraphs of G in six squares. Thus every $K_{3,3}$ subgraph of G that links γ, links γ in four squares which all share a common edge.

Let G_0 be a subgraph of G isomorphic to $K_{4,4}$ containing J. Label the two set of vertices of G_0 by {1,3,5,7} and {2,4,6,8}, and the remaining vertices of G label 9 and 0 appropriately.

Case 1(a): For every square α in G_0 and edge e of α, there is no pair of α-opposite subgraphs of G_0 such that e is the common edge of the four squares linking γ in both subgraphs.

Suppose without loss of generality, that γ links 123456 in the four squares 1234, 1236, 1254, and 1256 (common edge $\overline{12}$), and links the 1234-opposite subgraph 123478 in four squares with a different common edge. Since γ links 1234 in 123478 the common edge must be an edge of 1234. There are two different ways this can happen, either the common edge is adjacent to $\overline{12}$ or not. First consider the linking where the common edge ($\overline{34}$) is not adjacent to the first common edge ($\overline{12}$). Then γ links 123478 in the four squares 1234, 1438, 3274, and 3478. So γ links the two disjoint squares 1256 and 3478 which are both in G_0. Call 1256, L_o and 3478, L_1 . Now $(L \setminus J) \cup L_o \cup L_1$ is a non-split n-component link, and $(L \setminus J) \cup L_o$ is a non-split $(n-1)$-component link.

Next suppose that γ links 123478 in the four squares with common edge adjacent to $\overline{12}$, say $\overline{14}$. So γ links 1234, 1274, 1438 and 1478 in 123478. We see as follows that this forces γ to link twelve squares in G_0. The simple closed curve γ links the three squares 1234, 1438 and 1254 in 123458. These squares have a single edge $\overline{14}$ in common, so γ must also link 1458. Similarly, γ links 1234, 1236, and 1274 in 123476 and therefore the additional square 1276. Next, γ links the two squares 1236 and 1256 in 123658. These squares have two edges in common $\overline{12}$ and $\overline{16}$. However $\overline{12}$ cannot be the common edge because the squares 1258 and 1238 are in the subgraph 123458, and are not among the four squares γ links in this subgraph. Thus, $\overline{16}$ is the common edge for 123658. So γ also links 1638 and 1658. Finally, γ links the three squares 1438, 1458 and 1638 in 143678, and therefore also links 1678. So we have found twelve squares of G_0, 1234, 1236, 1254, 1256, 1274, 1276, 1438, 1638, 1458, 1658, 1478, and 1678 that γ links. By inspection the $K_{3,3}$ subgraphs of G_0: 125478, 125678, and 145678 as well as the above mentioned $K_{3,3}$ subgraphs of G_0: 123456, 123476, 123458, 123478, 123658, and 143678 each contains exactly four of the above mentioned twelve squares (that γ links) with a common edge. Together these nine $K_{3,3}$ subgraphs of G_0 contain all of the squares in G_0. Thus these are the only squares in G_0 that γ links, because in this way we see that it does not link any other square in G_0. If none of these twelve squares in G_0 is contained in a link in G_0, then there is not a pair of squares in G_0 that will form a non-split n-component link together with L.

Let the subgraph of G defined by the vertices 14365870 be G_1. The simple closed curve γ links 143658 in the four squares 1438, 1458, 1638, and 1658. Notice the common edge is $\overline{18}$. The simple closed curve γ also links the 1438-opposite subgraph 143870. There are three different ways this can happen. If γ links 143870 with common edge $\overline{18}$ then we are in **case 1(b)**. If γ links 143870 with the common

edge $\overline{34}$ then there are two disjoint squares 1658 and 4307 linking γ as described two paragraphs above. If γ links 143870 in four squares with common edge $\overline{14}$, then it links twelve squares in G_1, which we can find as before: 1438, 1458, 1478, 1638, 1658, 1678, 1430, 1450, 1470, 1630, 1650, and 1670. Again there is a possibility that the links in G_1 do not contain any of these squares. Finally, consider the subgraph 12365870. The simple closed curve γ links 123658 in the four squares 1236, 1256, 1638, and 1658 (common edge $\overline{16}$) and it links the 1236-opposite subgraph 123670 in the four squares with the same common edge $\overline{16}$ (i.e. 1236, 1276, 1630 and 1670). So this puts us in **case 1(b)**.

Case 1(b): There is a square α in G_0 and an edge e of α such that G_0 has a pair of α-opposite subgraphs with e as the common edge of the four squares linking γ in both.

Assume, without loss of generality that γ links 1234, and 123456 and 123478 are a pair of α-opposite subgraphs that γ links with the same common edge say $\overline{12}$. So γ links the square 1234, 1236, 1254, and 1256 in 123456,and 1234, 1238, 1274, and 1278 in 123478. This forces γ to link 1258, because it links the three squares 1234, 1254, and 1238 in 123458. Similarly it forces γ to link 1276 in the subgraph 123476. Hence γ links the squares 1234, 1254, 1236, 1256, 1238, 1258, 1274, 1276, and 1278. Thus γ links every square in G_0 containing $\overline{12}$. The graph G_0 is isomorphic to $K_{4,4}$, so each edge is contained in one component of a link of two components [6]. Take the non-split link $L_o \cup L_1$ in G such that L_o contains the edge $\overline{12}$. Thus L_o links γ, and hence $(L \setminus J) \cup L_o \cup L_1$ a non-split n-component link in $K_{2n+1,2n+1}$ and $(L \setminus J) \cup L_o$ is a non-split $(n-1)$-component link.

Case 2: The curve γ links some $K_{3,3}$ subgraph of G in six squares.

Let G_0 be a subgraph of G isomorphic to $K_{4,4}$ that contains some $K_{3,3}$ subgraph that γ links in six squares. Label the two set of vertices of G_0 by {1,3,5,7} and {2,4,6,8}.

Case 2(a): There is no square α linking γ in G_0, such that the pair of α-opposite subgraphs in G both link γ in six squares.

Without loss of generality γ links 1234 and links a subgraph of G_0 isomorphic to $K_{3,3}$ containing 1234 in six squares and the 1234-opposite subgraph in G isomorphic to $K_{3,3}$ in four squares. Without loss of generality, γ links 123456 in four squares, say, 1234, 1236, 1254, and 1256, and links 123478 in six squares. By Lemma 4, there are three mutually nonadjacent edges in 123478 which each appear in precisely two of six squares linking γ. Any square in 123478 will contain at least one of the edges in a set of three mutually nonadjacent edges. So each square in a set of six squares which links γ must contain exactly one of these three edges. Since the set of six squares linking γ in 123478 includes 1234, precisely one of the mutually nonadjacent edges must be an edge of 1234. Thus the mutually nonadjacent edges are determined by taking an edge of 1234 and two other nonadjacent edges of 123478 not in the square 1234. However, once the edge of 1234 is chosen, it determines the remaining pair of mutually nonadjacent edges in 123478 which are not in 1234. Since there are four choices for the edge in 1234, there are four ways γ can link 123478 in six squares given that 1234 links γ. These four possibilities are listed below with their three mutually nonadjacent edges that appear exactly twice:

possibility 1	possibility 2	possibility 3	possibility 4
$\overline{34}, \overline{27}, \overline{18}$	$\overline{23}, \overline{18}, \overline{47}$	$\overline{14}, \overline{27}, \overline{38}$	$\overline{12}, \overline{38}, \overline{47}$

So the sets of squares that γ links in 123478 are as follows:

possibility 1	possibility 2	possibility 3	possibility 4
1234	1234	1234	1234
1274	1274	3274	3274
1238	1438	1238	1438
1478	1278	3478	3278
3278	3278	1278	1278
3478	3478	1478	1478

Possibilities 1, 2, and 3 contain 3478. In these possibilities, γ links two disjoint squares in G_0, namely 3478 and 1256. Thus $(L \setminus J) \cup 3478 \cup 1258$ is a non-split n-component link, and $(L \setminus J) \cup 3478$ is a non-split $(n-1)$-component link. So we are done.

Now consider the possibility 4. At the beginning of this case we assumed that γ is linked to 1254, and since we are in possibility 4, γ also links 1278 and 1478. There is no edge that appears in all three of these squares so by Lemma 4, γ links 125478 in six squares. Also, γ links 123678 in 1236, 3278, and 1278. So, by Lemma 4, γ also links 123678 in six squares. Thus γ links both 125478 and 123678 in six squares including 1278. Since 124578 and 123678 are 1278-opposite graphs, this violates the hypothesis of this case.

Case 2(b): There is a square α in G_0 linking γ such that the pair of α-opposite subgraphs in G_0 both link γ in six squares.

Without loss of generality, $\alpha = 1234$, and the pair of 1234-opposite subgraphs 123456 and 123478 both link γ in six squares. Thus there are three mutually nonadjacent edges of 123456, each of which appears in two of the six squares that γ links, and each of the other edges of 123456 appears in three of the squares that γ links. As we saw in **Case 2(a)**, precisely one of these three mutually nonadjacent edges appears in each of the squares of 123456 linking γ. Since γ links 1234, without loss of generality $\overline{12}$ is one of these edges, and hence $\overline{34}$ is not one of these edges. Thus $\overline{56}$ is also not one of these edges. Hence $\overline{56}$ appears in three of the squares linking γ in 123456. Similarly, $\overline{78}$ appears in three of the squares linking γ in 123478. In 123456 there are four squares containing $\overline{56}$; they are of the form $ab56$ with $a \in \{1,3\}$ and $b \in \{2,4\}$. There are also four squares containing $\overline{78}$ in 123478 and they are of the form $cd78$ with $c \in \{1,3\}$ and $d \in \{2,4\}$. Since three of each of these sets of squares link γ there must be two disjoint squares L_o and L_1 in G_0 that each link γ. Now $(L \setminus J) \cup L_o \cup L_1$ is a non-split n-component link, and $(L \setminus J) \cup L_o$ is a non-split $(n-1)$-component link. $\square$

Remark 1. *For $K_{r,s}$ to be intrinsically n-linked it must contain n disjoint simple closed curves. The smallest simple closed curve in a complete bipartite graph is a square. A square contains four vertices, two from each of the sets of vertices. So the smallest complete bipartite graph that could be intrinsically n-linked is $K_{2n,2n}$. Here we have shown that $K_{2n+1,2n+1}$ is intrinsically n-linked. If $n = 2$ it is known that $K_{4,4}$ is intrinsically linked. For $n > 2$ there remain two graphs for which it is not known whether $K_{2n,2n}$ or $K_{2n,2n+1}$ are intrinsically n-linked. Thus it is not known whether $K_{2n+1,2n+1}$ is the smallest intrinsically n-linked bipartite graph.*

Corollary 1. *Let $n > 1$, and $K_{2n+1,2n+1}$ be embedded in $\mathbb{R}^3$, every edge of $K_{2n+1,2n+1}$ is contained in a non-split n-component link.*

Proof. Suppose there is an edge $e \in K_{2n+1,2n+1}$ that is not contained in a non-split n-component link. Then choose a subgraph of $K_{2n+1,2n+1}$ isomorphic to $K_{4,4}$ that contains the edge e. Since every edge of $K_{4,4}$ is contained in a non-split 2-component link [6]. The edge e is contained in a non-split 2-component link, $L_o \cup L_1$ in $K_{4,4}$. Without loss of generality assume $e \in L_o$, and notice that both L_o and L_1 are squares. Now assume that $n > 2$. Next choose a $K_{7,7}$ subgraph of $K_{2n+1,2n+1}$, which contains this $K_{4,4}$. Then using the construction in the proof of the Theorem 1, with $J = L_1$ and $\gamma = L_o$ we obtain a non-split 3-component link in $K_{7,7}$ containing L_o and hence the edge e. Thus we now assume that $n > 3$. The resulting link is $L_o \cup L_2 \cup L_3$, where either L_2 and L_3 link L_o or L_2 links both L_o and L_3. Now consider a $K_{9,9}$ subgraph of $K_{2n+1,2n+1}$, which contains $K_{7,7}$. In the first case, where L_2 and L_3 link L_o, take $J = L_3$ and $\gamma = L_o$ to get a non-split 4-component link in $K_{9,9}$ containing L_o. In the second case, where L_2 links L_o and L_3, take $J = L_3$ and $\gamma = L_2$ to get a non-split 4-component link that contains L_o. In either case, the resulting link contains L_o and therefore it contains e. We can continue in this way, at each stage choosing J to be a square constructed in the previous step and γ to be a square linking J. The subsequent larger non-split link will always contain the edge e. □

With minor changes to the proof of Theorem 1 and Corollary 1 we can prove a slightly stronger result about tripartite graphs, as follows:

Theorem 2. *Given $n > 1$, every embedding of the complete tripartite graph $K_{2n,2n,1}$ into $\mathbb{R}^3$ contains a non-split n-component link.*

Corollary 2. *Let $K_{2n,2n,1}$ be embedded in $\mathbb{R}^3$, every edge of $K_{2n,2n,1}$ is contained in a non-split n-component link.*

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